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HOCHSCHILD COHOMOLOGY OF KLEIN SURFACES

Frédéric BUTIN[1]

Abstract
Given a mechanical system (M, $\mathcal{F}(M)$), where M is a Poisson manifold and $\mathcal{F}(M)$ the algebra of regular functions on M, it is important to be able to quantize it, in order to obtain more precise results than through classical mechanics. An available method is the deformation quantization, which consists in constructing a star-product on the algebra of formal power series $\mathcal{F}(M)[[\hbar]]$. A first step toward study of star-products is the calculation of Hochschild cohomology of $\mathcal{F}(M)$.
The aim of this article is to determine this Hochschild cohomology in the case of singular curves of the plane — so we rediscover, by a different way, a result proved by Fronsdal and make it more precise — and in the case of Klein surfaces. The use of a complex suggested by Kontsevich and the help of Gröbner bases allow us to solve the problem.

Résumé
Etant donné un système physique (M, $\mathcal{F}(M)$), où M est une variété de Poisson et $\mathcal{F}(M)$ l'algèbre des fonctions régulières sur M, il est important de pouvoir le quantifier pour obtenir des résultats plus corrects que ceux donnés par la mécanique classique. Une solution est fournie par la quantification par déformation qui consiste à construire un star-produit sur l'algèbre des séries formelles $\mathcal{F}(M)[[\hbar]]$. Un premier pas vers l'étude des star-produits est le calcul de la cohomologie de Hochschild de $\mathcal{F}(M)$.
Le but de l'article est de déterminer cette cohomologie de Hochschild dans le cas des courbes singulières du plane — on précise ainsi, par une démarche différente, un résultat démontré par Fronsdal — et dans le cas des surfaces de Klein. L'utilisation d'un complexe proposé par Kontsevich et l'emploi des bases de Gröbner permettent de résoudre le problème.

Keywords
cohomology ; Hochschild ; Klein surfaces ; Gröbner bases ; quantization ; star-products.

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1 Introduction

1.1 Deformation quantization

We consider a mechanical system given by a Poisson manifold M, endowed with a Poisson bracket $\{\cdot\}$. In classical mechanics, we study the (commutative) algebra $\mathcal{F}(M)$ of regular functions (ie, for example, $\mathcal{C}^\infty$, holomorphic or polynomial) on M, that is to say the observables of the classical system. But quantum mechanics, where the physical system is described by a (non commutative) algebra of operators on a Hilbert space, gives more correct results than its classical analogous. Hence the importance to get a quantum description of the classical system $(M,\, \mathcal{F}(M))$: such an operation is called a quantization. One option is geometric quantization, which allows to construct in an explicit way a Hilbert space and an algebra of operators on this space. This very interesting method presents the drawback of being seldom applicable. That's why have been introduced other methods such as asymptotic quantization and deformation quantization. The latter, described in 1978 by F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer in the article [BFFLS78], is a good alternative : instead of an algebra of operators on a Hilbert space, a formal deformation of $\mathcal{F}(M)$. This is given by the algebra of formal power series $\mathcal{F}(M)[[\hbar]]$, endowed with some associative, but not commutative, star-product :

$$f * g = \sum_{j=0}^{\infty} m_j(f,\, g)\hbar^j \tag{1}$$

where the maps m_j are bilinear and where $m_0(f,\, g) = fg$. Then quantization is given by the map $f \mapsto \widehat{f}$, where the operator $\widehat{f}$ satisfies $\widehat{f}(g) = f * g$.
In which cases does a Poisson manifold admit such a quantization ? The answer was given by Kontsevich in his article [K97] : in fact he constructed a star-product on every Poisson manifold. Besides he proved that if M is a smooth manifold, then the equivalence classes of formal deformations of the zero Poisson bracket are in bijection with equivalence classes of star-products. Moreover, as a consequence of Hochschild-Kostant-Rosenberg theorem, every abelian star-product is equivalent to a trivial one.
In the case where M is a singular algebraic variety, say

$$M = \{\mathbf{z} \in \mathbb{C}^n \ /\ f(\mathbf{z}) = 0\}$$

with $n = 2$ or 3, where f belongs to $\mathbb{C}[\mathbf{z}]$ — and this is the case which we shall study — we shall consider the algebra of functions on M, ie the quotient algebra $\mathbb{C}[\mathbf{z}] \,/\, \langle f \rangle$. So the above mentioned result is no more valid. However, the deformations of the algebra $\mathcal{F}(M)$, defined by the formula (1), are always classified by the Hochschild cohomology of $\mathcal{F}(M)$, and we are led to the study of the Hochschild cohomology of $\mathbb{C}[\mathbf{z}] \,/\, \langle f \rangle$.

1.2 Cohomologies and quotients of polynomial algebras

We shall now consider $R = \mathbb{C}[z_1,\ \ldots,\ z_n] = \mathbb{C}[\mathbf{z}]$ the polynomial algebra with complex coefficients and n variables. We also fix m elements $f_1,\ \ldots,\ f_m$ of R, and we define the quotient algebra $A := R \,/\, \langle f_1,\ \ldots,\ f_m \rangle = \mathbb{C}[z_1,\ \ldots,\ z_n] \,/\, \langle f_1,\ \ldots,\ f_m \rangle$.

Several articles were devoted to the study of particular cases, for Hochschild as well as for Poisson cohomology :

C. Roger and P. Vanhaecke, in the article [RV02], consider the case where $n = 2$ and $m = 1$, and where f_1 is an homogeneous polynomial. They calculate Poisson cohomology in terms of the number of irreducible components of the singular locus $\{\mathbf{z} \in \mathbb{C}^2 \ / \ f_1(\mathbf{z}) = 0\}$ (in this case, we have a symplectic structure outside the singular locus).

M. Van den Bergh and A. Pichereau, in the articles [VB94], [P05] and [P06], are interested in the case where $n = 3$ and $m = 1$, and where f_1 is a weighted homogeneous polynomial with an isolated singularity at the origin. They compute the Poisson homology and cohomology, which in particular can be expressed in terms of the Milnor number of the space $\mathbb{C}[z_1,\, z_2,\, z_3] \ / \ \langle \partial_{z_1} f_1,\, \partial_{z_2} f_1,\, \partial_{z_3} f_1 \rangle$ (the definition of this number is given in [AVGZ86]).

Once more in the case where $n = 3$ and $m = 1$, in the article [AL98], J. Alev and T. Lambre compare the Poisson homology in 0 degree of Klein surfaces with the Hochschild homology in 0 degree of $A_1(\mathbb{C})^G$, where $A_1(\mathbb{C})$ is the Weyl algebra and G the group associated to the Klein surface. We shall give more details about those surfaces in section 4.1.

C. Fronsdal studies in the article [FK07] Hochschild homology and cohomology in two particular cases : the case where $n = 1$ and $m = 1$, and the case where $n = 2$ and $m = 1$. Besides, the appendix of this article gives another way to calculate the Hochschild cohomology in the more general case of complete intersections.

In this paper, we propose to calculate the Hochschild cohomology in two particularly important cases : the case of the singular curves of the plane, with polynomials f_1 which correspond to their normal forms (this case already held C. Fronsdal's attention) ; and the case of Klein surfaces $\mathcal{X}_\Gamma$ which are the quotients $\mathbb{C}^2 \,/\, \Gamma$, where Γ is a finite subgroup of $\mathbf{SL}_2\mathbb{C}$ (this case corresponds to $n = 3$ and $m = 1$). These latter have been the subject of many works ; their link with the finite subgroups of $\mathbf{SL}_2\mathbb{C}$, with the Platonic polyhedra, and with McKay correspondence explain this large interest. Moreover, the preprojective algebras, to which the article [CBH98] is devoted, constitute a family of deformations of the Klein surfaces, parametered by the group which is associated to them : this fact justifies once again the calculation of their cohomology.
The main result of the article is given by two propositions :

Proposition 1
Let be a singular curve of the plane, defined by a polynomial $f_1 \in \mathbb{C}[\mathbf{z}]$, of the type A_k, D_k or E_k. Then $H^0 \simeq \mathbb{C}[\mathbf{z}] \,/\, \langle f_1 \rangle$, $H^1 \simeq \mathbb{C}[\mathbf{z}] \,/\, \langle f_1 \rangle \,\oplus\, \mathbb{C}^k$, and for all $j \geq 2$, $H^j \simeq \mathbb{C}^k$.

Proposition 2
Let Γ be a finite subgroup of $\mathbf{SL}_2\mathbb{C}$ and $f_1 \in \mathbb{C}[\mathbf{z}]$ such that $\mathbb{C}[x,\, y]^\Gamma \simeq \mathbb{C}[\mathbf{z}] \,/\, \langle f_1 \rangle$. Then $H^0 \simeq \mathbb{C}[\mathbf{z}] \,/\, \langle f_1 \rangle$, $H^1 \simeq \nabla f_1 \wedge (\mathbb{C}[\mathbf{z}] \,/\, \langle f_1 \rangle)^3 \,\oplus\, \mathbb{C}^\mu$, $H^2 \simeq \mathbb{C}[\mathbf{z}] \,/\, \langle f_1 \rangle \,\oplus\, \mathbb{C}^\mu$, and for all $j \geq 3$, $H^j \simeq \mathbb{C}^\mu$, where μ is the Milnor number of $\mathcal{X}_\Gamma$.

For explicit computations, we shall use methods suggested by M. Kontsevich in the appendix of [FK07] ; we will develop a little bit that method.
We will first study the case of singular curves of the plane in section 3 : we will use this method to rediscover the result that C. Fronsdal proved by direct calculations. Then we will refine it by determining the dimensions of the cohomology spaces by means of the multivariate division and of the Gröbner bases. Next, in section 4, we will consider the case of Klein surfaces $\mathcal{X}_\Gamma$. We will first prove that H^0 identifies with the space of polynomial functions on the singular surface $\mathcal{X}_\Gamma$. We will then prove that H^1 and H^2 are infinite-dimensional. We will also determine, for j greater or equal to 3, the dimension of H^j, by showing that it is equal to the Milnor number of the surface $\mathcal{X}_\Gamma$.
Now, section 1.3 will recall important classical results about deformations.

1.3 Hochschild cohomology and deformations of algebras

• Given an associative $\mathbb{C}-$algebra, denoted by A, the Hochschild complex of A is the following

$$C^0(A) \xrightarrow{d_0} C^1(A) \xrightarrow{d_1} C^2(A) \xrightarrow{d_2} C^3(A) \xrightarrow{d_3} C^4(A) \xrightarrow{d_4} \cdots$$

where the space $C^p(A)$ of $p-$cochains is defined by $C^p(A) = 0$ for $p \in -\mathbb{N}^*$, $C^0(A) = A$ and $\forall\ p \in \mathbb{N}^*,\ C^p(A) = L(A^{\otimes p},\ A)$, where $L(A^{\otimes p},\ A)$ defines the space of $\mathbb{C}-$linear maps from $A^{\otimes p}$ to A, and where differential $d = \bigoplus_{i=0}^{\infty} d_p$ is given by

$$\forall\ f \in C^p(A),\ d_p\, f(a_0, \ldots,\ a_p) = a_0 f(a_1, \ldots,\ a_p) - \sum_{i=0}^{p-1} (-1)^i f(a_0, \ldots,\ a_i a_{i+1}, \ldots,\ a_p) + (-1)^{p-1} f(a_0, \ldots,\ a_{p-1}) a_p.$$

We can write it in terms of Gerstenhaber bracket $[\cdot]_G$ as follows :

$$d_p f = (-1)^{p+1} [\mu,\ f]_G,$$

where μ is the product of the algebra A, and $[\cdot]_G$ the Gerstenhaber brachet.
Then we define the Hochschild cohomology of A as the cohomology of the Hochschild complex associated to A. We note $HH^0(A) = \mathrm{Ker}\ d_0$ and $\forall\ p \in \mathbb{N}^*,\ HH^p(A) = \mathrm{Ker}\ d_p\ /\ \mathrm{Im}\ d_{p-1}$.

• We denote by $\mathbb{C}[[\hbar]]$ (resp. $A[[\hbar]]$) the algebra of formal power series in the parameter $\hbar$, with coefficients in $\mathbb{C}$ (resp. A). A deformation of the algebra A is defined as a map m from $A[[\hbar]] \times A[[\hbar]]$ to $A[[\hbar]]$ which is $\mathbb{C}[[\hbar]]-$bilinear and such that

$$\begin{array}{l} \forall\ (s,\ t) \in A[[\hbar]]^2,\ \ m(s,\ t) \equiv st \mod \hbar A[[\hbar]], \\ \forall\ (s,\ t,\ u) \in A[[\hbar]]^3,\ \ m(s,\ m(t,\ u)) = m(m(s,\ t),\ u). \end{array}$$

This means that there exists a sequence of bilinear maps m_j from $A \times A$ to A of which the first term m_0 is the product of A and such that

$$\begin{array}{l} \forall\ (a,\ b) \in A^2,\ m(a,\ b) = \displaystyle\sum_{j=0}^{\infty} m_j(a,\ b)\hbar^j, \\ \forall\ n \in \mathbb{N},\ \displaystyle\sum_{i+j=n} m_i(a,\ m_j(b,\ c)) = \sum_{i+j=n} m_i(m_j(a,\ b),\ c),\ \text{that is to say} \sum_{i+j=n} m_i \bullet m_j = 0, \end{array}$$

by using the notation $\bullet$ for the Gerstenhaber product.
We talk about deformation of order p if the previous formula is satisfied (only) for $n \leq p$.
Two deformations m and m' of A are called equivalent if there exists a $\mathbb{C}[[\hbar]]-$automorphism of $A[[\hbar]]$, denoted by φ, such that

$$\begin{array}{l} \forall\ (s,\ t) \in A[[\hbar]]^2,\ \varphi(m(s,\ t)) = m'(\varphi(s),\ \varphi(t)) \\ \forall\ s \in A[\hbar],\ \varphi(s) \equiv s \mod \hbar A[[\hbar]], \end{array}$$

that is to say if there exists a sequence of linear maps φ_j from A to A of which the first term φ_0 is the identity of A and such that

$$\begin{array}{l} \forall\ a \in A,\ \varphi(a) = \displaystyle\sum_{j=0}^{\infty} \varphi_j(a)\hbar^j, \\ \forall\ n \in \mathbb{N},\ \displaystyle\sum_{i+j=n} \varphi_i(m_j(a,\ b)) = \sum_{i+j+k=n} m'_i(\varphi_j(a),\ \varphi_k(b)). \end{array}$$

• One of the advantages of Hochschild cohomology is its classification of deformations of the algebra A. In fact, if $\pi \in C^2(A)$, we can construct a first order deformation m of A such that $m_1 = \pi$ if and only if $\pi \in \mathrm{Ker}\, d_2$. Moreover, two first order deformations are equivalent if and only if their difference is an

element of $\operatorname{Im} d_1$. So the set of classes of first order deformations is in bijection with $HH^2(A)$.
If m is a deformation of order p, then we can extend m to a deformation of order $p+1$ if and only if there exists m_{p+1} such that

$$\forall\ (a,\, b,\, c) \in A^3,\ \underbrace{\sum_{i=1}^{p} (m_i(a,\, m_{p+1-i}(b,\, c)) - m_i(m_{p+1-i}(a,\, b),\, c))}_{\omega_p(a,\, b,\, c)} = -d_2\, m_{p+1}(a,\, b,\, c),$$

$$\text{ie} \sum_{i=1}^{p} m_i \bullet m_{p+1-i} = d_2 m_{p+1}.$$

But ω_p belongs to $\operatorname{Ker} d_3$, as an easy computation shows, so $HH^3(A)$ contains the obstructions to extend a deformation of order p to a deformation of order $p+1$.

2 Presentation of the Koszul complex

We recall in this section some results about the Koszul complex used in the following and which are given in the appendix of the article [FK07].

2.1 Kontsevich theorem and notations

• As indicated in section 1.2, we consider $R = \mathbb{C}[\mathbf{z}]$ and $(f_1,\, \dots,\, f_m) \in R^m$, and we note $A = R\ /\ \langle f_1,\ \dots,\ f_m \rangle$. We suppose that there is a *complete intersection*, ie the dimension of the solution set of the system $\{f_1 = \cdots = f_m = 0\}$ is $n - m$.

• We also define the super-commutative super-algebra $\widetilde{A} = R \otimes \bigwedge\{\alpha_j,\ j = 1 \dots m\} = \mathbb{C}[z_1,\ \dots,\ z_n,\ \alpha_1,\ \dots,\ \alpha_m]$. Then we introduce $\eta_i = \frac{\partial}{\partial z_i}$ and $b_j = \frac{\partial}{\partial \alpha_j}$.
We denote the even variables by roman letters and the odd variables by greek letters.

• We consider the differential graded algebra

$$\widetilde{T} = A[\eta_1,\ \dots,\ \eta_n,\ b_1,\ \dots,\ b_m] = \frac{\mathbb{C}[z_1,\ \dots,\ z_n]}{\langle f_1,\ \dots,\ f_m \rangle}[\eta_1,\ \dots,\ \eta_n,\ b_1,\ \dots,\ b_m],$$

endowed with the differential

$$\boxed{d_{\widetilde{T}} = \sum_{j=1}^{n} \sum_{i=1}^{m} \frac{\partial f_i}{\partial z_j} b_i \frac{\partial}{\partial \eta_j}}$$

and the Hodge grading, defined by $deg(z_i) = 0,\ deg(\eta_i) = 1,\ deg(\alpha_j) = -1,\ deg(b_j) = 2$.

Then we can set forth the main theorem which allows us the calculation of the Hochschild cohomology :

Theorem 3 *(Kontsevich)*
Under the previous assumptions, the Hochschild cohomology of A is isomorphic to the cohomology of the complex $(\widetilde{T},\ d_{\widetilde{T}})$ associated to the differential graded algebra $\widetilde{T}$.

Remark 4
Theorem 3 can be seen as a generalization of Hochschild-Kostant-Rosenberg theorem to the case of non-smooth spaces.

• There is no element of negative degree. So the complex is as follows :

$$\widetilde{T}(0) \xrightarrow{\tilde{0}} \widetilde{T}(1) \xrightarrow{d^{(1)}_{\widetilde{T}}} \widetilde{T}(2) \xrightarrow{d^{(2)}_{\widetilde{T}}} \widetilde{T}(3) \xrightarrow{d^{(3)}_{\widetilde{T}}} \widetilde{T}(4) \xrightarrow{d^{(4)}_{\widetilde{T}}} \cdots$$

For each degree p, we choose a basis $\mathcal{B}_p$ of $\widetilde{T}(p)$. For example for $p = 0 \dots 3$, let's take :

$\widetilde{T}(0) = A$
$\widetilde{T}(1) = A\eta_1 \oplus \cdots \oplus A\eta_n$
$\widetilde{T}(2) = Ab_1 \oplus \cdots \oplus Ab_m \oplus \bigoplus_{i<j} A\,\eta_i\eta_j$
$\widetilde{T}(3) = \bigoplus_{\substack{i=1\dots m\\ j=1\dots n}} A\,b_i\eta_j \oplus \bigoplus_{i<j<k} A\,\eta_i\eta_j\eta_k$

Then we can make matrices $Mat_{\mathcal{B}_p,\,\mathcal{B}_{p+1}}(d^{(p)}_{\widetilde{T}})$ explicit.

• We note $p\ :\ \mathbb{C}[\mathbf{z}] \to A = \mathbb{C}[\mathbf{z}]/\langle f_1, \dots,\ f_m\rangle$ the canonical projection.
For each ideal J of $\mathbb{C}[\mathbf{z}]$, we denote by J_A the image of this ideal by the canonical projection.
Similarly if $(g_1, \dots,\ g_r) \in A^r$ we denote by $\langle g_1, \dots,\ g_r\rangle_A$ the ideal of A generated by $(g_1, \dots,\ g_r)$.
Besides, if $g \in \mathbb{C}[\mathbf{z}]$, and if J is an ideal of $\mathbb{C}[\mathbf{z}]$, then we note

$$Ann_J(g) := \{h \in \mathbb{C}[\mathbf{z}]\ /\ hg = 0 \mod J\}.$$

In particular, g doesn't divide 0 in $\mathbb{C}[\mathbf{z}]/J$ if and only if $Ann_J(g) = J$.
Finally, let's denote by ∇g the gradient of a polynomial $g \in \mathbb{C}[\mathbf{z}]$.

2.2 Particular case where $n = 1$ and $m = 1$

• In the case where $n = 1$ and $m = 1$, according to what we have seen, we have for $p \in \mathbb{N}^*$,

$$\boxed{\widetilde{T}(2p) = Ab_1^p} \text{ and } \boxed{\widetilde{T}(2p+1) = Ab_1^p\eta_1.}$$

We deduce
$H^0 = A$, $H^1 = \{g_1\eta_1\ /\ g_1 \in A \text{ and } g_1\,\partial_{z_1} f_1 = 0\}$
and $\forall\ p \in \mathbb{N}^*$, $H^{2p} = \dfrac{Ab_1^p}{\{g_1(\partial_{z_1} f_1)b_1^p\ /\ g_1 \in A\}}$, and $H^{2p+1} = \{g_1\, b_1^p\eta_1\ /\ g_1 \in A \text{ and } g_1\,\partial_{z_1} f_1 = 0\}$.

• Now if $f_1 = z_1^k$, then

$H^0 = A = \mathbb{C}[z_1]\ /\ \langle z_1^k\rangle \simeq \mathbb{C}^{k-1}$
$H^1 = \{g_1\eta_1\ /\ g_1 \in A \text{ and } kg_1z_1^{k-1} = 0\} \simeq \mathbb{C}^{k-1}$
and $\forall\ p \in \mathbb{N}^*$, $H^{2p} = \dfrac{Ab_1^p}{\{g_1(kz_1^{k-1})b_1^p\ /\ g_1 \in A\}} \simeq \mathbb{C}^{k-1}$
and $H^{2p+1} = \{g_1\, b_1^p\eta_1\ /\ g_1 \in A \text{ and } kg_1z_1^{k-1} = 0\} \simeq \mathbb{C}^{k-1}$.

3 Case $n = 2,\ m = 1$. — singular curves of the plane

3.1 Description of the cohomology spaces

With the help of theorem 3 we calculate the Hochschild cohomology of A. We begin by making cochains and differentials explicit.

• The various spaces of the complex are given by

$$\begin{array}{l|l} \widetilde{T}(0) = A & \widetilde{T}(5) = Ab_1^2\eta_1 \oplus Ab_1^2\eta_2 \\ \widetilde{T}(1) = A\eta_1 \oplus A\eta_2 & \widetilde{T}(6) = Ab_1^3 \oplus Ab_1^2\eta_1\eta_2 \\ \widetilde{T}(2) = Ab_1 \oplus A\eta_1\eta_2 & \widetilde{T}(7) = Ab_1^3\eta_1 \oplus Ab_1^3\eta_2 \\ \widetilde{T}(3) = Ab_1\eta_1 \oplus Ab_1\eta_2 & \widetilde{T}(8) = Ab_1^4 \oplus Ab_1^3\eta_1\eta_2 \\ \widetilde{T}(4) = Ab_1^2 \oplus Ab_1\eta_1\eta_2 & \widetilde{T}(9) = Ab_1^4\eta_1 \oplus Ab_1^4\eta_2, \end{array}$$

ie, in the generic case, $\boxed{\widetilde{T}(2p) = Ab_1^p \oplus Ab_1^{p-1}\eta_1\eta_2}$ and $\boxed{\widetilde{T}(2p+1) = Ab_1^p\eta_1 \oplus Ab_1^p\eta_2}$.

We have $\frac{\partial}{\partial \eta_k}(\eta_k \wedge \eta_l) = 1 \wedge \eta_l = -\eta_l \wedge 1$, hence $d_{\widetilde{T}}^{(2)}(\eta_k\eta_l) = -\frac{\partial f_1}{\partial z_k} b_1\eta_l + \frac{\partial f_1}{\partial z_l} b_1\eta_k$.
From now on, we note $\frac{\partial}{\partial_{z_j}} = \partial_{z_j} = \partial_j$.

The matrices of $d_{\widetilde{T}}$ are therefore given by

$$Mat_{\mathcal{B}_{2p},\mathcal{B}_{2p+1}}(d_{\widetilde{T}}^{(2p)}) = \begin{pmatrix} 0 & \partial_2 f_1 \\ 0 & -\partial_1 f_1 \end{pmatrix}$$
$$Mat_{\mathcal{B}_{2p+1},\mathcal{B}_{2p+2}}(d_{\widetilde{T}}^{(2p+1)}) = \begin{pmatrix} \partial_1 f_1 & \partial_2 f_1 \\ 0 & 0 \end{pmatrix}.$$

• We deduce a simpler expression for the cohomology spaces :

$H^0 = A$
$H^1 = \{g_1\eta_1 + g_2\eta_2 \ / \ (g_1,\, g_2) \in A^2 \text{ and } g_1\,\partial_1 f_1 + g_2\,\partial_2 f_1 = 0\} \simeq \left\{ \mathbf{g} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in A^2 \ / \ \mathbf{g}\cdot\nabla f_1 = 0 \right\}$
$\forall\, p \in \mathbb{N}^*$,

$$\begin{aligned} H^{2p} &= \frac{\{g_1 b_1^p + g_2 b_1^{p-1}\eta_1\eta_2 \ / \ (g_1,\, g_2)\in A^2 \text{ and } g_2\,\partial_1 f_1 = g_2\,\partial_2 f_1 = 0\}}{\{(g_1\,\partial_1 f_1 + g_2\,\partial_2 f_1) b_1^p \ / \ (g_1,\, g_2)\in A^2\}} \simeq \frac{\left\{ \mathbf{g} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in A^2 \ / \ g_2\,\partial_1 f_1 = g_2\,\partial_2 f_1 = 0 \right\}}{\left\{ \begin{pmatrix} \mathbf{g}\cdot\nabla f_1 \\ 0 \end{pmatrix} \ / \ \mathbf{g}\in A^2 \right\}} \\ &\simeq \frac{A}{\langle \partial_1 f_1,\ \partial_2 f_1 \rangle_A} \oplus \{g \in A \ / \ g\,\partial_1 f_1 = g\,\partial_2 f_1 = 0\} \end{aligned}$$

$$H^{2p+1} = \frac{\{g_1 b_1^p\eta_1 + g_2 b_1^p\eta_2 \ / \ (g_1,\, g_2)\in A^2 \text{ and } g_1\,\partial_1 f_1 + g_2\,\partial_2 f_1 = 0\}}{\{g_2(\partial_2 f_1 b_1^p\eta_1 - \partial_1 f_1 b_1^p\eta_2) \ / \ g_2\in A\}} \simeq \frac{\left\{ \mathbf{g} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in A^2 \ / \ \mathbf{g}\cdot\nabla f_1 = 0 \right\}}{\left\{ g_2 \begin{pmatrix} \partial_2 f_1 \\ -\partial_1 f_1 \end{pmatrix} \ / \ g_2\in A \right\}}.$$

It remains to determine these spaces more explicitly. It will be done in the two following sections.

3.2 Explicit calculations in the particular case where f_1 has separate variables

In this section, we consider the polynomial $f_1 = a_1 z_1^k + a_2 z_2^l$, with $2 \le l \le k$ and $(a_1,\, a_2) \in (\mathbb{C}^*)^2$.
The partial derivative of f_1 are $\partial_1 f_1 = k a_1 z_1^{k-1}$ and $\partial_2 f_1 = l a_2 z_2^{l-1}$.

• We already have

$$H^0 = \mathbb{C}[z_1,\, z_2]/\langle a_1 z_1^k + a_2 z_2^l \rangle.$$

• Besides, as f_1 is of homogeneous weight, Euler's formula gives $\dfrac{1}{k} x_1 \partial_1 f_1 + \dfrac{1}{l} x_2 \partial_2 f_1 = f_1$. So we have the inclusion $\langle f_1 \rangle \subset \langle \partial_1 f_1,\, \partial_2 f_1 \rangle$, hence $\frac{A}{\langle \partial_{z_1} f_1,\ \partial_{z_2} f_1 \rangle_A} \simeq \frac{\mathbb{C}[z_1,\, z_2]}{\langle \partial_{z_1} f_1,\ \partial_{z_2} f_1 \rangle} \simeq Vect\left(z_1^i z_2^j \ / \ i \in [\![0,\, k-2]\!],\ j \in [\![0,\, l-2]\!] \right)$.
But $\partial_1 f_1$ and f_1 are relatively prime, just as $\partial_2 f_1$ and f_1 are, hence if $g \in A$ satisfies $g\partial_1 f_1 = 0 \mod \langle f_1 \rangle$, then $g \in \langle f_1 \rangle$, ie g is zero in A.
So,

$$H^{2p} \simeq Vect\left(z_1^i z_2^j \ / \ i \in [\![0,\, k-2]\!],\ j \in [\![0,\, l-2]\!] \right) \simeq \mathbb{C}^{(k-1)(l-1)}.$$

• Now we determine the set $\left\{ \mathbf{g} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in A^2 \;/\; \mathbf{g} \cdot \nabla f_1 = 0 \right\}$:
First we have $\langle f_1,\, \partial_1 f_1 \rangle = \langle a_1 z_1^k + a_2 z_2^l,\, z_1^{k-1} \rangle = \langle z_2^l,\, z_1^{k-1} \rangle$. So the only monomials which are not in this ideal are the elements $z_1^i z_2^j$ with $i \in [\![0,\, k-2]\!]$ and $j \in [\![0,\, l-1]\!]$.
Every polynomial $P \in \mathbb{C}[\mathbf{z}]$ can be written in the form

$$P = \alpha f_1 + \beta \partial_1 f_1 + \sum_{\substack{i=0\ldots k-2 \\ j=0\ldots l-1}} a_{ij} z_1^i z_2^j.$$

The polynomials $P \in \mathbb{C}[\mathbf{z}]$ such that $P \partial_2 f_1 \in \langle f_1,\, \partial_1 f_1 \rangle$ are hence the elements

$$P = \alpha f_1 + \beta \partial_1 f_1 + \sum_{\substack{i=0\ldots k-2 \\ j=1\ldots l-1}} a_{ij} z_1^i z_2^j.$$

So we have calculated $Ann_{\langle f_1,\, \partial_1 f_1 \rangle}(\partial_2 f_1)$.
The equation

$$\mathbf{g} \cdot \nabla f_1 = 0 \mod \langle f_1 \rangle \tag{2}$$

yields

$$g_2 \partial_2 f_1 = 0 \mod \langle f_1,\, \partial_1 f_1 \rangle, \tag{3}$$

ie $g_2 \in Ann_{\langle f_1,\, \partial_1 f_1 \rangle}(\partial_2 f_1)$, ie again

$$g_2 = \alpha f_1 + \beta \partial_1 f_1 + \sum_{\substack{i=0\ldots k-2 \\ j=1\ldots l-1}} a_{ij} z_1^i z_2^j, \tag{4}$$

with $(\alpha,\, \beta) \in \mathbb{C}[\mathbf{z}]^2$.
It follows that

$$g_1 \partial_1 f_1 + \alpha f_1 \partial_2 f_1 + \beta \partial_1 f_1 \partial_2 f_1 + \sum_{\substack{i=0\ldots k-2 \\ j=1\ldots l-1}} a_{ij} z_1^i z_2^j \partial_2 f_1 \in \langle f_1 \rangle. \tag{5}$$

And, from the equality $z_2 \partial_2 f_1 = l f_1 - \frac{l}{k} z_1 \partial_1 f_1$, one deduces :

$$\partial_1 f_1 \left(g_1 + \beta \partial_2 f_1 - \frac{l}{k} \sum_{\substack{i=0\ldots k-2 \\ j=1\ldots l-1}} a_{ij} z_1^{i+1} z_2^{j-1} \right) \in \langle f_1 \rangle. \tag{6}$$

$$\text{ie } g_1 = -\beta \partial_2 f_1 + \frac{l}{k} \sum_{\substack{i=0\ldots k-2 \\ j=1\ldots l-1}} a_{ij} z_1^{i+1} z_2^{j-1} + \delta f_1,$$

with $\delta \in \mathbb{C}[\mathbf{z}]$.
Then we verify that the elements g_1 and g_2 obtained in this way are indeed solutions of the equation (2).
Finally, we have :

$$\left\{ \mathbf{g} \in A^2 \;/\; \mathbf{g} \cdot \nabla f_1 = 0 \right\} = \left\{ \begin{pmatrix} \alpha \\ \delta \end{pmatrix} f_1 - \beta \begin{pmatrix} \partial_2 f_1 \\ -\partial_1 f_1 \end{pmatrix} + \sum_{\substack{i=0\ldots k-2 \\ j=1\ldots l-1}} a_{ij} z_1^i z_2^{j-1} \begin{pmatrix} \frac{l}{k} z_1 \\ z_2 \end{pmatrix} \Big/ \ (\alpha,\, \beta,\, \delta) \in \mathbb{C}[\mathbf{z}]^3 \text{ and } a_{ij} \in \mathbb{C} \right\}.$$

We immediately deduce the cohomology spaces of odd degree :

$$\begin{array}{rcl} \forall\, p \geq 1,\ H^{2p+1} & \simeq & \mathbb{C}^{(k-1)(l-1)} \\ H^1 & \simeq & \mathbb{C}^{(k-1)(l-1)} \oplus \mathbb{C}[z_1,\, z_2]/\langle a_1 z_1^k + a_2 z_2^l \rangle. \end{array}$$

Remark 5
We obtain in particular the cohomology for the cases where $f_1 = z_1^{k+1} + z_2^2$, $f_1 = z_1^3 + z_2^4$ *and* $f_1 = z_1^3 + z_2^5$. *These cases correspond respectively to the weight homogeneous functions of types* A_k, E_6 *and* E_8 *given in [AVGZ86] p. 181.*

The table below summarizes the results obtained for the three particular cases we have just obtained :

	H^0	H^1	H^{2p}	H^{2p+1}
A_k	$\mathbb{C}[\mathbf{z}]\ /\ \langle z_1^{k+1} + z_2^2\rangle$	$\mathbb{C}[\mathbf{z}]\ /\ \langle z_1^{k+1} + z_2^2\rangle \oplus \mathbb{C}^k$	$\mathbb{C}^k$	$\mathbb{C}^k$
E_6	$\mathbb{C}[\mathbf{z}]\ /\ \langle z_1^3 + z_2^4\rangle$	$\mathbb{C}[\mathbf{z}]\ /\ \langle z_1^3 + z_2^4\rangle \oplus \mathbb{C}^6$	$\mathbb{C}^6$	$\mathbb{C}^6$
E_8	$\mathbb{C}[\mathbf{z}]\ /\ \langle z_1^3 + z_2^5\rangle$	$\mathbb{C}[\mathbf{z}]\ /\ \langle z_1^3 + z_2^5\rangle \oplus \mathbb{C}^8$	$\mathbb{C}^8$	$\mathbb{C}^8$

The cases where $f_1 = z_1^2 z_2 + z_2^{k-1}$ and $f_1 = z_1^3 + z_1 z_2^3$, ie respectively D_k and E_7, will be studied in the next section.

3.3 Explicit calculations for D_k and E_7

To study these particular cases, we use the following result about Gröbner bases (theorem 7). First, recall the definition of a Gröbner basis. For $g \in \mathbb{C}[\mathbf{z}]$, we denote by $lt(g)$ its leading term (for the lexicographic order). Given a non trivial ideal J of $\mathbb{C}[\mathbf{z}]$, a Gröbner basis of J is a finite subset G_J of $J\backslash\{0\}$ such that for all $f \in J\backslash\{0\}$, there exists $g \in G_J$ such that $lt(g)$ divides $lt(f)$.

Definition 6
1. Let J *be a non trivial ideal of* $\mathbb{C}[\mathbf{z}]$ *and* $G_J := [g_1, \ldots, g_r]$ *a Gröbner basis of* J. *A polynomial* $p \in \mathbb{C}[\mathbf{z}]$ *is reduced relatively to* G_J *if it is zero or if none of the terms of* p *is divisible by the leading term* $lt(g_j)$ *of one of the elements of* G_J.
2. The set of the G_J-*standard terms is the set of all monomials of* $\mathbb{C}[\mathbf{z}]$ *except the set of the leading terms* $lt(f)$ *of the polynomials* $f \in J\backslash\{0\}$.

Theorem 7 *(Macaulay)*
The set of the G_J-*standard terms forms a basis of the quotient vector space* $\mathbb{C}[\mathbf{z}]\ /\ J$.

3.3.1 Case of $f_1 = z_1^2 z_2 + z_2^{k-1}$, ie D_k

Here we have $f_1 = z_1^2 z_2 + z_2^{k-1}$, $\partial_1 f_1 = 2z_1 z_2$ and $\partial_2 f_1 = z_1^2 + (k-1)z_2^{k-2}$.
A Gröbner basis of the ideal $\langle f_1,\, \partial_2 f_1\rangle$ is $B := [b_1,\, b_2] = [z_1^2 + (k-1)z_2^{k-2},\, z_2^{k-1}]$.
So the set of the standard terms is$\{z_1^i z_2^j\ /\ i \in \{0,\, 1\}$ and $j \in [\![0,\, k-2]\!]\}$.
Then we can solve the equation $p\,\partial_1 f_1 = 0$ in $\mathbb{C}[\mathbf{z}]\ /\ \langle f_1,\, \partial_2 f_1\rangle$.
In fact, by writing $p := \displaystyle\sum_{i=0,1 j=0\ldots k-2} a_{ij} z_1^i z_2^j$, the equation becomes

$$q := \sum_{i=0,1 j=0\ldots k-2} a_{ij} z_1^{i+1} z_2^{j+1} \in \langle f_1,\, \partial_2 f_1\rangle.$$

So we look for the normal form of the element q modulo the ideal $\langle f_1,\, \partial_2 f_1\rangle$.
The multivariate division of q by B is $q = q_1 b_1 + q_2 b_2 + r$ with $r = \sum_{j=0}^{k-3} a_{0,j} z_1 z_2^{j+1}$.
Thus the solution is

$$p = a_{0,k-2} z_2^{k-2} + \sum_{j=0}^{k-2} a_{1,j} z_1 z_2^j.$$

But the equation

$$\mathbf{g} \cdot \nabla f_1 = 0 \mod \langle f_1\rangle \tag{7}$$

yields

$$g_1 \partial_1 f_1 = 0 \mod \langle f_1,\, \partial_2 f_1 \rangle, \tag{8}$$

ie

$$g_1 = \alpha f_1 + \beta \partial_2 f_1 + a z_2^{k-2} + \sum_{j=0}^{k-2} b_j z_1 z_2^j, \tag{9}$$

with $(\alpha,\, \beta) \in \mathbb{C}[\mathbf{z}]^2$ and $a,\, b_j \in \mathbb{C}$.
Hence

$$g_2 \partial_2 f_1 + \beta \partial_1 f_1 \partial_2 f_1 + a z_2^{k-2} \partial_1 f_1 + \sum_{j=0}^{k-2} b_j z_1 z_2^j \partial_1 f_1 \in \langle f_1 \rangle. \tag{10}$$

And with the equalities,

$$z_2^{k-1} = \frac{1}{2-k}(f_1 - z_2 \partial_2 f_1) = -\frac{1}{2-k} z_2 \partial_2 f_1 \mod \langle f_1 \rangle, \text{ and } \frac{k-2}{2} z_1 \partial_1 f_1 + z_2 \partial_2 f_1 = (k-1) f_1 \text{ (Euler)},$$

we obtain

$$\partial_2 f_1 \left(g_2 + \beta \partial_1 f_1 - \frac{2a}{2-k} z_1 z_2 + \sum_{j=0}^{k-2} b_j \frac{2}{2-k} z_2^{j+1} \right) \in \langle f_1 \rangle. \tag{11}$$

$$\text{ie, } g_2 = -\beta \partial_1 f_1 + \frac{2a}{2-k} z_1 z_2 - \sum_{j=0}^{k-2} b_j \frac{2}{2-k} z_2^{j+1} + \delta f_1,$$

with $\delta \in \mathbb{C}[\mathbf{z}]$. So

$$\left\{ \mathbf{g} \in A^2 \;/\; \mathbf{g} \cdot \nabla f_1 = 0 \right\} = \left\{ \begin{pmatrix} \alpha \\ \delta \end{pmatrix} f_1 + \beta \begin{pmatrix} \partial_2 f_1 \\ -\partial_1 f_1 \end{pmatrix} + \begin{pmatrix} z_2^{k-2} \\ \frac{2}{2-k} z_1 z_2 \end{pmatrix} + \sum_{j=0}^{k-2} b_j z_2^j \begin{pmatrix} z_1 \\ -\frac{2}{2-k} z_2 \end{pmatrix} \;\Big/\; (\alpha,\, \beta,\, \delta) \in \mathbb{C}[\mathbf{z}]^3 \text{ and } a,\, b_j \in \mathbb{C} \right\}.$$

On the other hand, a Gröbner basis of $\langle \partial_1 f_1,\, \partial_2 f_1 \rangle$ is $[z_1^2 + (k-1) z_2^{k-2},\, z_1 z_2,\, z_2^{k-1}]$,
thus $\mathbb{C}[\mathbf{z}] \;/\; \langle \partial_1 f_1,\, \partial_2 f_1 \rangle \simeq Vect\left(z_1,\, 1,\, z_2\, \ldots,\, z_2^{k-2}\right)$.

Let's summarize,

$$\boxed{\begin{array}{l} H^0 = \mathbb{C}[\mathbf{z}] \;/\; \langle z_1^2 z_2 + z_2^{k-1} \rangle \\ H^1 \simeq \mathbb{C}[\mathbf{z}] \;/\; \langle z_1^2 z_2 + z_2^{k-1} \rangle \oplus \mathbb{C}^k \\ H^{2p} \simeq \mathbb{C}^k \\ H^{2p+1} \simeq \mathbb{C}^k. \end{array}}$$

3.3.2 Case of $f_1 = z_1^3 + z_1 z_2^3$, ie E_7

Here we have $\partial_1 f_1 = 3z_1^2 + z_2^3$ and $\partial_2 f_1 = 3 z_1 z_2^2$.
A Gröbner basis of the ideal $\langle f_1,\, \partial_1 f_1 \rangle$ is $[3z_1^2 + z_2^3,\, z_1 z_2^3,\, z_2^6]$, and a Gröbner basis of $\langle \partial_1 f_1,\, \partial_2 f_1 \rangle$ is $[3z_1^2 + z_2^3,\, z_1 z_2^2,\, z_2^5]$.
By an analogous proof, we obtain :

$$\boxed{\begin{array}{l} H^0 = \mathbb{C}[\mathbf{z}] \;/\; \langle z_1^3 + z_1 z_2^3 \rangle \\ H^1 \simeq \mathbb{C}[\mathbf{z}] \;/\; \langle z_1^3 + z_1 z_2^3 \rangle \oplus \mathbb{C}^7 \\ H^{2p} \simeq \mathbb{C}^7 \\ H^{2p+1} \simeq \mathbb{C}^7. \end{array}}$$

4 Case $n = 3,\ m = 1$. — Klein surfaces

4.1 Klein surfaces

Given a finite group G acting on $\mathbb{C}^n$, we associate to it, according to Erlangen program of Klein, the quotient space $\mathbb{C}^n/G$, ie the space whose points are the orbits under the action of G ; it is an algebraic variety, and the polynomial functions on this variety are the polynomial functions on $\mathbb{C}^n$ which are $G-$invariant. In the case of $\mathbf{SL}_2\mathbb{C}$, invariant theory allows us to associate a polynomial to any finite subgroup, as explained in proposition 9. Thus, to every finite subgroup of $\mathbf{SL}_2\mathbb{C}$ are associated the zero set of this polynomial ; it is an algebraic variety, called Klein surface.
In this section we recall some results about this surfaces. See the references [S77] and [CCK99] for more details.

Proposition 8
Every finite subgroup of $\mathbf{SL}_2\mathbb{C}$ is conjugate to one of the following groups :
- A_n *(cyclic),* $n \geq 1$ $(|A_n| = n)$
- D_n *(dihedral),* $n \geq 1$ $(|D_n| = 4n)$
- E_6 *(tetrahedral)* $(|E_6| = 24)$
- E_7 *(octahedral)* $(|E_7| = 48)$
- E_8 *(icosahedral)* $(|E_8| = 120)$.

Proposition 9
Let G be one of the groups of the preceding list. The ring of invariants is the following :

$$\mathbb{C}[x,\, y]^G = \mathbb{C}[e_1,\, e_2,\, e_3] = \mathbb{C}[e_1,\, e_2] \oplus e_3\mathbb{C}[e_1,\, e_2] \simeq \mathbb{C}[z_1,\, z_2,\, z_3]/\langle f_1\rangle,$$

where the invariants e_j are homogeneous polynomials, with e_1 and e_2 algebraically independent, and where f_1 is a weighted homogeneous polynomial with an isolated singularity at the origin.
These polynomials are given in the following table.

G	$e_1,\ e_2,\ e_3$	f_1	$\mathbb{C}[z_1,\, z_2,\, z_3]/\langle\partial_1 f_1,\ \partial_2 f_1,\ \partial_3 f_1\rangle$
A_n	$e_1 = x^n$ $e_2 = y^n$ $e_3 = xy$	$-n(z_1z_2 - z_3^n)$	$Vect(1,\, z_1,\, \ldots,\, z_3^{n-2})$ $\dim = n-1$
D_n	$e_1 = x^2y^2$ $e_2 = x^{2n} + (-1)^n y^{2n}$ $e_3 = x^{2n+1}y + (-1)^{n+1}xy^{2n+1}$	$\lambda_n(4z_1^{n+1}+(-1)^{n+1}z_1z_2^2+(-1)^n z_3^2)$ with $\lambda_n = 2n(-1)^{n+1}$	$Vect(1,\, z_2,\, z_1,\, \ldots,\, z_1^{n-1})$ $\dim = n+1$
E_6	$e_1 = x^5y - xy^5$ $e_2 = 14y^4x^4 + x^8 + y^8$ $e_3 = 33y^8x^4 - y^{12} + 33y^4x^8 - x^{12}$	$4(z_3^2 - z_2^3 + 108z_1^4)$	$Vect(1,\, z_2,\, z_1,\, z_1z_2,\, z_1^2,\, z_1^2z_2)$ $\dim = 6$
E_7	$e_1 = 14y^4x^4 + x^8 + y^8$ $e_2 = -3y^{10}x^2 + 6y^6x^6 - 3y^2x^{10}$ $e_3 = -34x^5y^{13} - yx^{17} + 34y^5x^{13} + xy^{17}$	$8(3z_3^2 - 12z_2^3 + z_2z_1^3)$	$Vect(1,\, z_2,\, z_2^2,\, z_1,\, z_1z_2,\, z_1z_2^2,\, z_1^2)$ $\dim = 7$
E_8	$e_1 = x^{11}y + 11x^6y^6 - xy^{11}$ $e_2 = x^{20} - 228x^{15}y^5 + 494x^{10}y^{10}$ $+228x^5y^{15} + y^{20}$ $e_3 = x^{30} + 522x^{25}y^5$ $-10\,005x^{20}y^{10} - 10\,005x^{10}y^{20}$ $-522x^5y^{25} + y^{30}$	$10(1\,728z_1^5 + z_2^3 - z_3^2)$	$Vect(z_1^i z_2^j)_{\substack{i=0\ldots 3,\\ j=0\ldots 1}}$ $\dim = 8$

We call Klein surface the algebraic hyper-surface defined by $\{\mathbf{z} \in \mathbb{C}^3 \ / \ f_1(\mathbf{z}) = 0\}$.

Theorem 10 *(Pichereau)*
Consider the Poisson bracket defined on $\mathbb{C}[z_1,\, z_1,\, z_3]$ by

$$\{\cdot\}_{f_1} = \partial_3 f_1\, \partial_1 \wedge \partial_2 + \partial_1 f_1\, \partial_2 \wedge \partial_3 + \partial_2 f_1\, \partial_3 \wedge \partial_1 = i(df_1)(\partial_1 \wedge \partial_2 \wedge \partial_3),$$

and denote by $HP^*_{f_1}$ (resp. $HP^{f_1}_*$) the Poisson cohomology (resp. homology) for this bracket. Under the previous assumptions, the Poisson cohomology $HP^*_{f_1}$ and the Poisson homology $HP^{f_1}_*$ of $(\mathbb{C}[z_1,\, z_1,\, z_3]/\langle f_1\rangle,\ \{\cdot\}_{f_1})$ is given by

$$\begin{array}{l} HP^0_{f_1} = \mathbb{C}, \quad HP^1_{f_1} \simeq HP^2_{f_1} = \{0\} \\ HP^{f_1}_0 \simeq HP^{f_1}_2 \simeq \mathbb{C}[z_1,\, z_2,\, z_3]/\langle \partial_1 f_1,\ \partial_2 f_1,\ \partial_3 f_1\rangle \\ \dim(HP^{f_1}_1) = \dim(HP^{f_1}_0) - 1 \\ HP^{f_1}_j = HP^j_{f_1} = \{0\} \ si\ j \geq 3. \end{array}$$

The algebra $\mathbb{C}[x,\, y]$ is a Poisson algebra for the standard symplectic bracket $\{\cdot\}_{std}$. As G is a subgroup of the symplectic group $\mathbf{Sp}_2\mathbb{C}$ (since $\mathbf{Sp}_2\mathbb{C} = \mathbf{SL}_2\mathbb{C}$), the invariant algebra $\mathbb{C}[x,\, y]^G$ is a Poisson subalgebra of $\mathbb{C}[x,\, y]$. Then the following proposition allows us to deduce from theorem 10 the Poisson cohomology and homology of $\mathbb{C}[x,\, y]^G$ for the standard symplectic bracket.

Proposition 11
The isomorphism of associative algebras

$$\begin{array}{rcl} \pi : (\mathbb{C}[x,\, y]^G,\ \{\cdot\}_{std}) & \rightarrow & (\mathbb{C}[z_1,\, z_1,\, z_3]/\langle f_1\rangle,\ \{\cdot\}_{f_1}) \\ e_j & \mapsto & \overline{z_j} \end{array}$$

is a Poisson isomorphism.

Subsequently, we will calculate the Hochschild cohomology of $\mathbb{C}[z_1,\, z_1,\, z_3]/\langle f_1\rangle$. Then we will deduce immediately the Hochschild cohomology of $\mathbb{C}[x,\, y]^G$, with the help of the isomorphism of associative algebras π.

4.2 Description of the cohomology spaces

• In this case, we change the ordering of the basis : we shall use $(\eta_1\eta_2,\ \eta_2\eta_3,\ \eta_3\eta_1)$ instead of $(\eta_1\eta_2,\ \eta_1\eta_3,\ \eta_2\eta_3)$. Then the different spaces of the complex are given by

$$\begin{array}{l} \widetilde{T}(0) = A \\ \widetilde{T}(1) = A\eta_1 \oplus A\eta_2 \oplus A\eta_3 \\ \widetilde{T}(2) = Ab_1 \oplus A\eta_1\eta_2 \oplus A\eta_2\eta_3 \oplus A\eta_3\eta_1 \\ \widetilde{T}(3) = Ab_1\eta_1 \oplus Ab_1\eta_2 \oplus Ab_1\eta_3 \oplus A\eta_1\eta_2\eta_3 \\ \widetilde{T}(4) = Ab_1^2 \oplus Ab_1\eta_1\eta_2 \oplus Ab_1\eta_2\eta_3 \oplus Ab_1\eta_3\eta_1 \\ \widetilde{T}(5) = Ab_1^2\eta_1 \oplus Ab_1^2\eta_2 \oplus Ab_1^2\eta_3 \oplus Ab_1\eta_1\eta_2\eta_3 \end{array}$$

ie, in the general case, for $p \in \mathbb{N}^*$, $\boxed{\widetilde{T}(2p) = Ab_1^p \oplus Ab_1^{p-1}\eta_1\eta_2 \oplus Ab_1^{p-1}\eta_2\eta_3 \oplus Ab_1^{p-1}\eta_3\eta_1}$
and $\boxed{\widetilde{T}(2p+1) = Ab_1^p\eta_1 \oplus Ab_1^p\eta_2 \oplus Ab_1^p\eta_3 \oplus Ab_1^{p-1}\eta_1\eta_2\eta_3}$.

We have $\frac{\partial}{\partial\eta_1}(\eta_1 \wedge \eta_2 \wedge \eta_3) = 1 \wedge \eta_2 \wedge \eta_3 = \eta_2 \wedge \eta_3 \wedge 1$, thus $d^{(3)}_{\widetilde{T}}(\eta_1\eta_2\eta_3) = \frac{\partial f_1}{\partial z_1} b_1\eta_2\eta_3 + \frac{\partial f_1}{\partial z_2} b_1\eta_3\eta_1 + \frac{\partial f_1}{\partial z_3} b_1\eta_1\eta_2$.

The matrices of $d_{\widetilde{T}}$ are therefore given by

$$
\begin{aligned}
Mat_{\mathcal{B}_1,\mathcal{B}_2}(d^{(1)}_{\widetilde{T}}) &= \begin{pmatrix} \partial_{z_1} f_1 & \partial_{z_2} f_1 & \partial_{z_3} f_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\forall\ p \in \mathbb{N}^*,\ Mat_{\mathcal{B}_{2p},\mathcal{B}_{2p+1}}(d^{(2p)}_{\widetilde{T}}) &= \begin{pmatrix} 0 & \partial_{z_2} f_1 & 0 & -\partial_{z_3} f_1 \\ 0 & -\partial_{z_1} f_1 & \partial_{z_3} f_1 & 0 \\ 0 & 0 & -\partial_{z_2} f_1 & \partial_{z_1} f_1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
\forall\ p \in \mathbb{N}^*,\ Mat_{\mathcal{B}_{2p+1},\mathcal{B}_{2p+2}}(d^{(2p+1)}_{\widetilde{T}}) &= \begin{pmatrix} \partial_{z_1} f_1 & \partial_{z_2} f_1 & \partial_{z_3} f_1 & 0 \\ 0 & 0 & 0 & \partial_{z_3} f_1 \\ 0 & 0 & 0 & \partial_{z_1} f_1 \\ 0 & 0 & 0 & \partial_{z_2} f_1 \end{pmatrix}.
\end{aligned}
$$

• We deduce

$H^0 = A$

$H^1 = \{g_1\eta_1 + g_2\eta_2 + g_3\eta_3 \ /\ (g_1,\ g_2,\ g_3) \in A^3 \text{ and } g_1\,\partial_{z_1} f_1 + g_2\,\partial_{z_2} f_1 + g_3\,\partial_{z_3} f_1 = 0\} \simeq \left\{ \mathbf{g} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} \in A^3 \ /\ \mathbf{g}\cdot\nabla f_1 = 0 \right\}$

$$
\begin{aligned}
H^2 &= \frac{\{g_0 b_1 + g_3\eta_1\eta_2 + g_1\eta_2\eta_3 + g_2\eta_3\eta_1 \ /\ (g_0, g_1, g_2\ g_3) \in A^4 \text{ and } g_3\,\partial_{z_2} f_1 - g_2\,\partial_{z_3} f_1 = g_1\,\partial_{z_3} f_1 - g_3\,\partial_{z_1} f_1 = g_2\,\partial_{z_1} f_1 - g_1\,\partial_{z_2} f_1 = 0\}}{\{(g_1\,\partial_{z_1} f_1 + g_2\,\partial_{z_2} f_1 + g_3\,\partial_{z_3} f_1) b_1 \ /\ (g_1, g_2, g_3) \in A^3\}} \\
&\simeq \frac{\left\{ \mathbf{g} = \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{pmatrix} \in A^4 \Big/ \nabla f_1 \wedge \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = 0 \right\}}{\left\{ \begin{pmatrix} \mathbf{g}\cdot\nabla f_1 \\ \mathbf{0}_{3,1} \end{pmatrix} \ /\ \mathbf{g} \in A^3 \right\}} \\
&\simeq \frac{A}{\langle \partial_{z_1} f_1,\ \partial_{z_2} f_1,\ \partial_{z_3} f_1 \rangle_A} \oplus \{\mathbf{g} \in A^3 \ /\ \nabla f_1 \wedge \mathbf{g} = 0\}
\end{aligned}
$$

$\forall\ p \geq 2,$

$$
\begin{aligned}
H^{2p} &= \frac{\left\{ g_0 b_1^p + g_3 b_1^{p-1}\eta_1\eta_2 + g_1 b_1^{p-1}\eta_2\eta_3 + g_2 b_1^{p-1}\eta_3\eta_1 \Big/ \begin{matrix} (g_0, g_1, g_2\ g_3) \in A^4 \text{ and} \\ g_3\,\partial_{z_2} f_1 - g_2\,\partial_{z_3} f_1 = g_1\,\partial_{z_3} f_1 - g_3\,\partial_{z_1} f_1 = g_2\,\partial_{z_1} f_1 - g_1\,\partial_{z_2} f_1 = 0 \end{matrix} \right\}}{\{(g_1\,\partial_{z_1} f_1 + g_2\,\partial_{z_2} f_1 + g_3\,\partial_{z_3} f_1) b_1^p + g_0(\partial_{z_3} f_1\ b_1^{p-1}\eta_1\eta_2 + \partial_{z_1} f_1\ b_1^{p-1}\eta_2\eta_3 + \partial_{z_2} f_1\ b_1^{p-1}\eta_3\eta_1) \ /\ (g_0, g_1, g_2, g_3) \in A^3\}} \\
&\simeq \frac{\left\{ \mathbf{g} = \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{pmatrix} \in A^4 \Big/ \nabla f_1 \wedge \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = 0 \right\}}{\left\{ \begin{pmatrix} \mathbf{g}\cdot\nabla f_1 \\ g_0\,\partial_{z_1} f_1 \\ g_0\,\partial_{z_2} f_1 \\ g_0\,\partial_{z_3} f_1 \end{pmatrix} \ /\ \mathbf{g} \in A^3 \text{ and } g_0 \in A \right\}} \\
&\simeq \frac{A}{\langle \partial_{z_1} f_1,\ \partial_{z_2} f_1,\ \partial_{z_3} f_1 \rangle_A} \oplus \frac{\{\mathbf{g} \in A^3 \ /\ \nabla f_1 \wedge \mathbf{g} = 0\}}{\{g\nabla f_1 \ /\ g \in A\}}
\end{aligned}
$$

$\forall\ p \in \mathbb{N}^*,$

$$
\begin{aligned}
H^{2p+1} &= \frac{\left\{ g_1 b_1^p\eta_1 + g_2 b_1^p\eta_2 + g_3 b_1^p\eta_3 + g_0 b_1^{p-1}\eta_1\eta_2\eta_3 \Big/ \begin{matrix} (g_0, g_1, g_2\ g_3) \in A^4 \text{ and } g_1\,\partial_{z_1} f_1 + g_2\,\partial_{z_2} f_1 + g_3\,\partial_{z_3} f_1 = 0 \\ g_0\,\partial_{z_3} f_1 = g_0\,\partial_{z_1} f_1 = g_0\,\partial_{z_2} f_1 = 0 \end{matrix} \right\}}{\{(g_3\,\partial_{z_2} f_1 - g_2\,\partial_{z_3} f_1) b_1^p\eta_1 + (g_1\,\partial_{z_3} f_1 - g_3\,\partial_{z_1} f_1) b_1^p\eta_2 + (g_2\,\partial_{z_1} f_1 - g_1\,\partial_{z_2} f_1) b_1^p\eta_3 \ /\ (g_1, g_2, g_3) \in A^3\}} \\
&\simeq \frac{\left\{ \mathbf{g} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_0 \end{pmatrix} \in A^4 \Big/ \nabla f_1 \cdot \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = 0 \text{ and } g_0\,\partial_{z_3} f_1 = g_0\,\partial_{z_1} f_1 = g_0\,\partial_{z_2} f_1 = 0 \right\}}{\left\{ \begin{pmatrix} \nabla f_1 \wedge \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} \\ 0 \end{pmatrix} \Big/ \mathbf{g} \in A^3 \right\}} \\
&\simeq \frac{\{\mathbf{g} \in A^3 \ /\ \nabla f_1 \cdot \mathbf{g} = 0\}}{\{\nabla f_1 \wedge \mathbf{g} \ /\ \mathbf{g} \in A^3\}} \oplus \{g \in A \ /\ g\,\partial_{z_3} f_1 = g\,\partial_{z_1} f_1 = g\,\partial_{z_2} f_1 = 0\}.
\end{aligned}
$$

The following section will allows us to make those various spaces more explicit.

4.3 Explicit calculations in the particular case where f_1 has separate variables

In this section, we consider the polynomial $f_1 = a_1 z_1^i + a_2 z_2^j + a_3 z_3^k$, with $2 \leq i \leq j \leq k$ and $a_j \in \mathbb{C}^*$.

Its partial derivative are $\partial_1 f_1 = i a_1 z_1^{i-1}$, $\partial_2 f_1 = j a_2 z_2^{j-1}$ and $\partial_3 f_1 = k a_3 z_3^{k-1}$.

• We have already

$$\boxed{H^0 = \mathbb{C}[z_1,\, z_2,\, z_3]/\langle a_1 z_1^i + a_2 z_2^j + a_3 z_3^k \rangle}.$$

• Moreover, as f_1 is weight homogeneous, Euler's formula gives $\frac{1}{i} z_1 \partial_1 f_1 + \frac{1}{j} z_2 \partial_2 f_1 + \frac{1}{k} z_3 \partial_3 f_1 = f_1$. So we have the inclusion $\langle f_1 \rangle \subset \langle \partial_1 f_1,\, \partial_2 f_1,\, \partial_3 f_1 \rangle$, thus

$$\frac{A}{\langle \partial_1 f_1,\, \partial_2 f_1,\, \partial_3 f_1 \rangle_A} \simeq \frac{\mathbb{C}[z_1,\, z_2,\, z_3]}{\langle \partial_1 f_1,\, \partial_2 f_1,\, \partial_3 f_1 \rangle} \simeq Vect\left(z_1^p z_2^q z_3^r \;/\; p \in [\![0,\, i-2]\!],\; q \in [\![0,\, j-2]\!],\; r \in [\![0,\, k-2]\!] \right).$$

Finally, as $\partial_1 f_1$ and f_1 are relatively prime, if $g \in A$ verifies $g \partial_1 f_1 = 0 \mod \langle f_1 \rangle$, then $g \in \langle f_1 \rangle$, ie g is zero in A.

• Now we determine the set $\left\{ \mathbf{g} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} \in A^3 \;/\; \mathbf{g} \cdot \nabla f_1 = 0 \right\}$:

First we have $\langle f_1,\, \partial_1 f_1,\, \partial_2 f_1 \rangle = \langle a_1 z_1^i + a_2 z_2^j + a_3 z_2^k,\, z_1^{i-1},\, z_2^{j-1} \rangle = \langle z_1^{i-1},\, z_2^{j-1} z_3^k \rangle$. Thus the only monomials which are not in this ideal are the elements $z_1^p z_2^q z_3^r$ with $p \in [\![0,\, i-2]\!]$, $q \in [\![0,\, j-2]\!]$ and $r \in [\![0,\, k-1]\!]$. So every polynomial $P \in \mathbb{C}[\mathbf{z}]$ can be written in the form :

$$P = \alpha f_1 + \beta \partial_1 f_1 + \gamma \partial_2 f_1 + \sum_{\substack{p=0\ldots i-2 \\ q=0\ldots j-2 \\ r=0\ldots k-1}} a_{pqr} z_1^p z_2^q z_3^r.$$

The polynomials $P \in \mathbb{C}[\mathbf{z}]$ such that $P \partial_3 f_1 \in \langle f_1,\, \partial_1 f_1,\, \partial_2 f_1 \rangle$ are therefore the following ones :

$$P = \alpha f_1 + \beta \partial_1 f_1 + \gamma \partial_2 f_1 + \sum_{\substack{p=0\ldots i-2 \\ q=0\ldots j-2 \\ r=1\ldots k-1}} a_{pqr} z_1^p z_2^q z_3^r.$$

So we have calculated $Ann_{\langle f_1,\, \partial_1 f_1,\, \partial_2 f_1 \rangle}(\partial_3 f_1)$.
The equation

$$\mathbf{g} \cdot \nabla f_1 = 0 \mod \langle f_1 \rangle \tag{12}$$

leads to $g_3 \in Ann_{\langle f_1,\, \partial_1 f_1,\, \partial_2 f_1 \rangle}(\partial_3 f_1)$, ie

$$g_3 = \alpha f_1 + \beta \partial_1 f_1 + \gamma \partial_2 f_1 + \sum_{\substack{p=0\ldots i-2 \\ q=0\ldots j-2 \\ r=1\ldots k-1}} a_{pqr} z_1^p z_2^q z_3^r, \tag{13}$$

with $(\alpha,\, \beta,\, \gamma) \in \mathbb{C}[\mathbf{z}]^3$.
Hence

$$g_2 \partial_2 f_1 + \gamma \partial_2 f_1 \partial_3 f_1 + \sum_{\substack{p=0\ldots i-2 \\ q=0\ldots j-2 \\ r=1\ldots k-1}} a_{pqr} z_1^p z_2^q z_3^r \partial_3 f_1 \in \langle f_1,\, \partial_1 f_1 \rangle. \tag{14}$$

Thus, according to Euler's formula,

$$\partial_2 f_1 \left(g_2 + \gamma \partial_3 f_1 - \frac{k}{j} \sum_{\substack{p=0\dots i-2 \\ q=0\dots j-2 \\ r=1\dots k-1}} a_{pqr} z_1^p z_2^{q+1} z_3^{r-1} \right) \in \langle f_1,\, \partial_1 f_1 \rangle. \tag{15}$$

Since $Ann_{\langle f_1,\, \partial_1 f_1 \rangle}(\partial_2 f_1) = \langle f_1,\, \partial_1 f_1 \rangle$, this equation is equivalent to

$$g_2 = -\gamma \partial_3 f_1 + \frac{k}{j} \sum_{\substack{p=0\dots i-2 \\ q=0\dots j-2 \\ r=1\dots k-1}} a_{pqr} z_1^p z_2^{q+1} z_3^{r-1} + \delta f_1 + \varepsilon \partial_1 f_1,$$

with $\delta,\, \varepsilon \in \mathbb{C}[\mathbf{z}]$. It follows that

$$g_1 \partial_1 f_1 + \beta \partial_1 f_1 \partial_3 f_1 + \varepsilon \partial_1 f_1 \partial_2 f_1 + \sum_{\substack{p=0\dots i-2 \\ q=0\dots j-2 \\ r=1\dots k-1}} a_{pqr} z_1^p z_2^q z_3^r \partial_3 f_1 + \frac{k}{j} \sum_{\substack{p=0\dots i-2 \\ q=0\dots j-2 \\ r=1\dots k-1}} a_{pqr} z_1^p z_2^{q+1} z_3^{r-1} \partial_3 f_1 \in \langle f_1 \rangle. \tag{16}$$

And, according to Euler's formula,

$$\partial_1 f_1 \left(g_1 + \beta \partial_3 f_1 + \varepsilon \partial_2 f_1 - \frac{k}{i} \sum_{\substack{p=0\dots i-2 \\ q=0\dots j-2 \\ r=1\dots k-1}} a_{pqr} z_1^{p+1} z_2^q z_3^{r-1} \right) \in \langle f_1 \rangle. \tag{17}$$

$$\text{ie } g_1 = -\beta \partial_3 f_1 - \varepsilon \partial_2 f_1 + \frac{k}{i} \sum_{\substack{p=0\dots i-2 \\ q=0\dots j-2 \\ r=1\dots k-1}} a_{pqr} z_1^{p+1} z_2^q z_3^{r-1} + \eta f_1, \tag{18}$$

with $\eta \in \mathbb{C}[\mathbf{z}]$. Finally

$$\left\{ \mathbf{g} \in A^3 \ / \ \mathbf{g} \cdot \nabla f_1 = 0 \right\} = \left\{ \begin{pmatrix} \eta \\ \delta \\ \alpha \end{pmatrix} f_1 + \nabla f_1 \wedge \begin{pmatrix} -\gamma \\ \beta \\ -\varepsilon \end{pmatrix} + \sum_{\substack{p=0\dots i-2 \\ q=0\dots j-2 \\ r=1\dots k-1}} a_{pqr} z_1^p z_2^q z_3^{r-1} \begin{pmatrix} \frac{k}{i} z_1 \\ \frac{k}{j} z_2 \\ z_3 \end{pmatrix} \Big/ \ (\alpha,\, \beta,\, \gamma,\, \delta,\, \varepsilon,\, \eta) \in A^6 \text{ and } a_{pqr} \in \mathbb{C} \right\}.$$

We deduce immediately the cohomology spaces of odd degrees :

$$\boxed{\begin{array}{rcl} \forall\, p \geq 1,\ H^{2p+1} & \simeq & \mathbb{C}^{(i-1)(j-1)(k-1)} \\ H^1 & \simeq & \nabla f_1 \wedge \left(\mathbb{C}[\mathbf{z}]/\langle f_1 \rangle\right)^3 \oplus \mathbb{C}^{(i-1)(j-1)(k-1)}. \end{array}}$$

Remark :
We also have $\nabla f_1 \wedge \left(\mathbb{C}[\mathbf{z}]/\langle f_1 \rangle\right)^3 \simeq \left(\mathbb{C}[\mathbf{z}]/\langle f_1 \rangle\right)^3 \ / \ \{\mathbf{g} \ / \ \nabla f_1 \wedge \mathbf{g} = 0\} = \left(\mathbb{C}[\mathbf{z}]/\langle f_1 \rangle\right)^3 \ / \ (\mathbb{C}[\mathbf{z}]/\langle f_1 \rangle) \nabla f_1$.
Moreover the map

$$\begin{array}{ccc} (\mathbb{C}[\mathbf{z}]/\langle f_1 \rangle)^2 & \rightarrow & \nabla f_1 \wedge \left(\mathbb{C}[\mathbf{z}]/\langle f_1 \rangle\right)^3 \\ \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} & \mapsto & \nabla f_1 \wedge \begin{pmatrix} g_1 \\ g_2 \\ 0 \end{pmatrix} \end{array}$$

is injective, thus $\boxed{\nabla f_1 \wedge \left(\mathbb{C}[\mathbf{z}]/\langle f_1 \rangle\right)^3 \text{ is infinite-dimensional}}$.

• It remains to determine the set $\left\{ \mathbf{g} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} \in A^3 \ / \ \nabla f_1 \wedge \mathbf{g} = 0 \right\}$:

Let $\mathbf{g} \in A^3$ be such that $\nabla f_1 \wedge \mathbf{g} = 0$. It means that, modulo $\langle f_1 \rangle$, $\mathbf{g}$ verifies the system $\left\{ \begin{array}{rcl} \partial_2 f_1\, g_3 - \partial_3 f_1\, g_2 & = & 0 \\ \partial_3 f_1\, g_1 - \partial_1 f_1\, g_3 & = & 0 \\ \partial_1 f_1\, g_2 - \partial_2 f_1\, g_1 & = & 0 \end{array} \right.$

The first equation gives, modulo $\langle f_1,\, \partial_2 f_1 \rangle$, $\partial_3 f_1\, g_2 = 0$.
Now $Ann_{\langle f_1,\, \partial_2 f_1 \rangle}(\partial_3 f_1) = \langle f_1,\, \partial_2 f_1 \rangle$, therefore $g_2 = \alpha f_1 + \beta \partial_2 f_1$. Hence

$$\partial_2 f_1 (g_3 - \beta \partial_3 f_1) = 0 \mod \langle f_1 \rangle,$$

ie $g_3 = \gamma f_1 + \beta \partial_3 f_1$.
Finally, we obtain

$$\partial_3 f_1 (g_1 - \beta \partial_1 f_1) = 0 \mod \langle f_1 \rangle,$$

ie $g_1 = \delta f_1 + \beta \partial_1 f_1$.
So, $\{\mathbf{g} \in A^3 \ / \ \nabla f_1 \wedge \mathbf{g} = 0\} = \left\{ f_1 \begin{pmatrix} \delta \\ \alpha \\ \gamma \end{pmatrix} + \beta \nabla f_1 \ / \ \alpha,\, \beta,\, \gamma,\, \delta \in A \right\}$.

We deduce the cohomology spaces of even degrees :

$$\boxed{\begin{array}{rcl} \forall\, p \geq 2,\ H^{2p} & \simeq & A \ / \ \langle \partial_1 f_1,\, \partial_2 f_1,\, \partial_3 f_1 \rangle \simeq \mathbb{C}[\mathbf{z}] \ / \ \langle z_1^{i-1},\, z_2^{j-1},\, z_3^{k-1} \rangle \\ & \simeq & Vect\, (z_1^p z_2^q z_3^r \ / \ p \in [\![0,\, i-2]\!],\ q \in [\![0,\, j-2]\!],\ r \in [\![0,\, k-2]\!]) \simeq \mathbb{C}^{(i-1)(j-1)(k-1)} \\ H^2 & \simeq & \{\beta \nabla f_1 \ / \ \beta \in A\} \oplus \mathbb{C}^{(i-1)(j-1)(k-1)} \simeq \mathbb{C}[\mathbf{z}] \ / \ \langle a_1 z_1^i + a_2 z_2^j + a_3 z_3^k \rangle \oplus \mathbb{C}^{(i-1)(j-1)(k-1)}. \end{array}}$$

Remark 12
In particular, we obtain the cohomology for the cases where $f_1 = z_1^2 + z_2^2 + z_3^{k+1}$, $f_1 = z_1^2 + z_2^3 + z_3^4$ *and* $f_1 = z_1^2 + z_2^3 + z_3^5$. *These cases correspond respectively to the types* A_k, E_6 *and* E_8 *of the Klein surfaces.*

The following table sums up the results of those three special cases :

	H^0	H^1	H^2	H^{2p}	H^{2p+1}
A_k	$\mathbb{C}[\mathbf{z}] \ / \ \langle z_1^2 + z_2^2 + z_3^{k+1} \rangle$	$\nabla f_1 \wedge (\mathbb{C}[\mathbf{z}] \ / \ \langle f_1 \rangle)^3 \oplus \mathbb{C}^k$	$\mathbb{C}[\mathbf{z}] \ / \ \langle z_1^2 + z_2^2 + z_3^{k+1} \rangle \oplus \mathbb{C}^k$	$\mathbb{C}^k$	$\mathbb{C}^k$
E_6	$\mathbb{C}[\mathbf{z}] \ / \ \langle z_1^2 + z_2^3 + z_3^4 \rangle$	$\nabla f_1 \wedge (\mathbb{C}[\mathbf{z}] \ / \ \langle f_1 \rangle)^3 \oplus \mathbb{C}^6$	$\mathbb{C}[\mathbf{z}] \ / \ \langle z_1^2 + z_2^3 + z_3^4 \rangle \oplus \mathbb{C}^6$	$\mathbb{C}^6$	$\mathbb{C}^6$
E_8	$\mathbb{C}[\mathbf{z}] \ / \ \langle z_1^2 + z_2^3 + z_3^5 \rangle$	$\nabla f_1 \wedge (\mathbb{C}[\mathbf{z}] \ / \ \langle f_1 \rangle)^3 \oplus \mathbb{C}^8$	$\mathbb{C}[\mathbf{z}] \ / \ \langle z_1^2 + z_2^3 + z_3^5 \rangle \oplus \mathbb{C}^8$	$\mathbb{C}^8$	$\mathbb{C}^8$

The cases where $f_1 = z_1^2 + z_2^2 z_3 + z_3^{k-1}$ and $f_1 = z_1^2 + z_2^3 + z_2 z_3^3$, ie respectively D_k and E_7 are studied in the following section.

4.4 Explicit calculations for D_k and E_7

4.4.1 Case of $f_1 = z_1^2 + z_2^2 z_3 + z_3^{k-1}$, ie D_k

In this section, we consider the polynomial $f_1 = z_1^2 + z_2^2 z_3 + z_3^{k-1}$.
Its partial derivatives are $\partial_1 f_1 = 2z_1$, $\partial_2 f_1 = 2z_2 z_3$ and $\partial_3 f_1 = z_2^2 + (k-1) z_3^{k-2}$.

• We already have

$$\boxed{H^0 = \mathbb{C}[\mathbf{z}] / \langle z_1^2 + z_2^2 z_3 + z_3^{k-1} \rangle.}$$

• Besides, since f_1 is of homogeneous weight, Euler's formula gives

$$\frac{k-1}{2} z_1\, \partial_1 f_1 + \frac{k-2}{2} z_2\, \partial_2 f_1 + z_3\, \partial_3 f_1 = (k-1) f_1. \tag{19}$$

Thus, we have the inclusion $\langle f_1\rangle \subset \langle \partial_1 f_1,\, \partial_2 f_1,\, \partial_3 f_1\rangle$.
Moreover, a Gröbner basis of $\langle \partial_1 f_1,\, \partial_2 f_1,\, \partial_3 f_1\rangle$ is $[z_3^{k-1},\ z_2z_3,\ z_2^2+(k-1)z_3^{k-2},\ z_1]$, therefore

$$\frac{A}{\langle \partial_1 f_1,\, \partial_2 f_1,\, \partial_3 f_1\rangle_A} \simeq \frac{\mathbb{C}[z_1,\, z_2,\, z_3]}{\langle \partial_1 f_1,\, \partial_2 f_1,\, \partial_3 f_1\rangle} \simeq Vect\left(z_2,\, 1,\, z_3, \ldots,\, z_3^{k-2}\right).$$

Finally, as $\partial_1 f_1$ are f_1 relatively prime, if $g \in A$ verifies $g\partial_1 f_1 = 0 \mod \langle f_1\rangle$, then $g \in \langle f_1\rangle$, ie g is zero in A, thus $\{g \in A \;/\; g\,\partial_{z_3} f_1 = g\,\partial_{z_1} f_1 = g\,\partial_{z_2} f_1 = 0\} = 0$.

• Now we determine the set $\left\{ \mathbf{g} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} \in A^3 \;/\; \mathbf{g}\cdot\nabla f_1 = 0 \right\}$:
A Gröbner basis of $\langle f_1,\, \partial_1 f_1,\, \partial_3 f_1\rangle$ is $[z_1,\ z_3^{k-1},\ z_2^2+(k-1)z_3^{k-2}]$, thus a basis of $\mathbb{C}[\mathbf{z}] \;/\; \langle f_1,\, \partial_1 f_1,\, \partial_3 f_1\rangle$ is $\{z_2^i z_3^j \;/\; i \in \{0,\, 1\},\ j \in [\![0,\, k-2]\!]\}$.

We have already solved the equation $p\,\partial_2 f_1 = 0$ in this space ; its solution is $p = a_{0,k-2}z_3^{k-2} + \sum_{j=0}^{k-2} a_{1,j} z_2 z_3^j$.

The equation

$$\mathbf{g}\cdot\nabla f_1 = 0 \mod \langle f_1\rangle \tag{20}$$

leads to

$$g_2\partial_2 f_1 = 0 \mod \langle f_1,\, \partial_1 f_1,\, \partial_3 f_1\rangle, \tag{21}$$

hence

$$g_2 = \alpha f_1 + \beta\partial_1 f_1 + \gamma\partial_3 f_1 + a z_3^{k-2} + \sum_{j=0}^{k-2} b_j z_2 z_3^j, \tag{22}$$

with $(\alpha,\, \beta,\, \gamma) \in \mathbb{C}[\mathbf{z}]^3$.
And

$$g_3\partial_3 f_1 + \gamma\partial_3 f_1\partial_2 f_1 + a z_3^{k-2}\partial_2 f_1 + \sum_{j=0}^{k-2} b_j z_2 z_3^j \partial_2 f_1 \in \langle f_1,\, \partial_1 f_1\rangle. \tag{23}$$

Now according to Euler's formula (19) and the equality

$$z_3^{k-1}z_2 = \frac{1}{2-k}\left(z_2 f_1 - z_2z_3\partial_3 f_1 - \frac{1}{2}z_2z_1\partial_1 f_1\right) = -\frac{1}{2-k}z_2z_3\partial_3 f_1 \mod \langle f_1,\, \partial_1 f_1\rangle, \tag{24}$$

the equation (23) becomes

$$\partial_3 f_1\left(g_3 + \gamma\partial_2 f_1 - \frac{2a}{2-k}z_2z_3 - \sum_{j=0}^{k-2} b_j \frac{2}{2-k} z_3^{j+1}\right) \in \langle f_1,\, \partial_1 f_1\rangle. \tag{25}$$

As $Ann_{\langle f_1,\, \partial_1 f_1\rangle}(\partial_3 f_1) = \langle f_1,\, \partial_1 f_1\rangle$, this equation is equivalent to

$$g_3 = -\gamma\partial_2 f_1 + \frac{2a}{2-k}z_2z_3 + \sum_{j=0}^{k-2} b_j \frac{2}{2-k} z_3^{j+1} + \delta f_1 + \varepsilon\partial_1 f_1,$$

with $\delta,\, \varepsilon \in \mathbb{C}[\mathbf{z}]$.
We find

$$g_1 = -\beta\partial_2 f_1 - \varepsilon\partial_3 f_1 + \sum_{j=0}^{k-2} b_j \frac{k-1}{k-2} z_1 + \frac{a}{2-k} z_2 z_1 + \eta f_1, \tag{26}$$

with $\eta \in \mathbb{C}[\mathbf{z}]$.

Finally, we have

$$\left\{\mathbf{g} \in A^3 \ / \ \mathbf{g} \cdot \nabla f_1 = 0\right\} = \left\{ \begin{pmatrix} \eta \\ \alpha \\ \delta \end{pmatrix} f_1 + \nabla f_1 \wedge \begin{pmatrix} \gamma \\ \varepsilon \\ -\beta \end{pmatrix} + \sum_{j=0}^{k-2} b_j \begin{pmatrix} \frac{k-1}{k-2} z_1 z_3^j \\ z_2 z_3^j \\ -\frac{2}{2-k} z_3^{j+1} \end{pmatrix} + a \begin{pmatrix} \frac{1}{2-k} z_2 z_1 \\ z_3^{k-2} \\ \frac{2a}{2-k} z_2 z_3 \end{pmatrix} \Big/ \ (\alpha,\, \beta,\, \gamma,\, \delta,\, \varepsilon,\, \eta) \in A^6 \text{ and } a,\, b_j \in \mathbb{C} \right\},$$

as well as cohomology spaces of odd degrees :

$$\boxed{\begin{array}{rcl} \forall\, p \geq 1,\ H^{2p+1} & \simeq & \mathbb{C}^k \\ H^1 & \simeq & \nabla f_1 \wedge \left(\mathbb{C}[\mathbf{z}]/\langle f_1 \rangle\right)^3 \oplus \mathbb{C}^k. \end{array}}$$

• To show $\{\mathbf{g} \in A^3 \ / \ \nabla f_1 \wedge \mathbf{g} = 0\} = \left\{f_1\, \mathbf{g} + \beta \nabla f_1 \ / \ \mathbf{g} \in A^3,\ \beta \in A\right\}$, we proceed as in the case of separate variables.
We deduce the cohomology spaces of even degrees :

$$\boxed{\begin{array}{rcl} \forall\, p \geq 2,\ H^{2p} & \simeq & A \ / \ \langle \partial_1 f_1,\, \partial_2 f_1,\, \partial_3 f_1 \rangle \simeq Vect\left(z_2,\, 1,\, z_3, \ldots,\, z_3^{k-2}\right) \simeq \mathbb{C}^k \\ H^2 & \simeq & \{\beta \nabla f_1 \ / \ \beta \in A\} \oplus \mathbb{C}^k \simeq \mathbb{C}[\mathbf{z}] \ / \ \langle z_1^2 + z_2^2 z_3 + z_3^{k-1} \rangle \oplus \mathbb{C}^k. \end{array}}$$

4.4.2 Case of $f_1 = z_1^2 + z_2^3 + z_2 z_3^3$, ie E_7

Here we have $\partial_1 f_1 = 2z_1,\ \partial_2 f_1 = 3z_2^2 + z_3^3$ and $\partial_3 f_1 = 3z_2 z_3^2$.
The proof is similar to the one of the previous cases.
A Gröbner basis of $\langle \partial_1 f_1,\, \partial_2 f_1,\, \partial_3 f_1 \rangle$ is $[z_3^5,\ z_2 z_3^2,\ 3z_2^2 + z_3^3,\ z_1]$.
Similarly, a Gröbner basis of $\langle f_1,\, \partial_1 f_1,\, \partial_2 f_1 \rangle$ is $[z_3^6,\ z_2 z_3^3,\ 3z_2^2 + z_3^3,\ z_1]$.
We obtain the following results :

$$\boxed{\begin{array}{rcl} \forall\, p \geq 1,\ H^{2p+1} & \simeq & \mathbb{C}^7 \\ H^1 & \simeq & \nabla f_1 \wedge \left(\mathbb{C}[\mathbf{z}]/\langle f_1 \rangle\right)^3 \oplus \mathbb{C}^7. \end{array}}$$

$$\boxed{\begin{array}{rcl} H^0 & = & \mathbb{C}[\mathbf{z}] \ / \ \langle z_1^2 + z_2^3 + z_2 z_3^3 \rangle \\ \forall\, p \geq 2,\ H^{2p} & \simeq & A \ / \ \langle \partial_1 f_1,\, \partial_2 f_1,\, \partial_3 f_1 \rangle \simeq Vect\left(z_2,\, z_2^2,\, 1,\, z_3,\, z_3^2,\, z_3^3,\, z_3^4\right) \simeq \mathbb{C}^7 \\ H^2 & \simeq & \{\beta \nabla f_1 \ / \ \beta \in A\} \oplus \mathbb{C}^k \simeq \mathbb{C}[\mathbf{z}] \ / \ \langle z_1^2 + z_2^3 + z_2 z_3^3 \rangle \oplus \mathbb{C}^7. \end{array}}$$

Remark 13
In all the previously studied cases, there exists a triple $(i,\, j,\, k)$ such that $\{i,\, j,\, k\} = \{1,\, 2,\, 3\}$, and such that the map

$$\begin{array}{rcl} \mathbb{C}[\mathbf{z}] \ / \ \langle \partial_1 f_1,\, \partial_2 f_1,\, \partial_3 f_1 \rangle & \to & \{\textit{Solutions in } \mathbb{C}[\mathbf{z}] \ / \ \langle f_1,\, \partial_j f_1,\, \partial_k f_1 \rangle \textit{ of the equation } g\, \partial_i f_1 = 0\} \\ P & \mapsto & z_i\, P \mod \langle f_1,\, \partial_j f_1,\, \partial_k f_1 \rangle \end{array}$$

is an isomorphism of vector spaces.

Références

[AL98] Jacques Alev, Thierry Lambre, *Comparaison de l'homologie de Hochschild et de l'homologie de Poisson pour une déformation des surfaces de Klein.* In Algebra and operator theory (Tashkent, 1997), pp. 25-38. Kluwer Acad. Publ., Dordrecht, 1998.

[AVGZ86] V. Arnold, A. Varchenko, S. Goussein-Zadé, *Singularités des applications différentiables*, première partie, éditions Mir, Moscou, 1986.

[BCKT05] Alain Bruguières, Alberto Cattaneo, Bernhard Keller, Charles Torossian, *Déformation, Quantification, Théorie de Lie*, Panoramas et Synthèses, SMF, 2005.

[BFFLS78] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, D. Sternheimer, *Deformation theory and quantization. I and II. Physical applications*, Ann. Physics **111**, no. 1, p. 61-110 and 111-151, 1978.

[CBH98] William Crawley-Boevey, Martin P. Holland, *Noncommutative Deformations of Kleinian Singularities*, Duke Mathematical Journal, Vol. 92, No. 3, 1998.

[CCK99] Li Chiang, Huah Chu, Ming-chang Kang, *Generation of Invariants*, Journal of Algebra, Volume 221, Issue 1, pp. 232-241, 1999.

[FK07] Christian Fronsdal, avec un appendice de Maxim Kontsevich, *Quantization on Curves*, Math-ph/0507021. Lett. Math. Phys. 79, pp. 109-129, 2007.

[GRS07] Laurent Guieu, Claude Roger, avec un appendice de Vlad Sergiescu, *L'Algèbre et le Groupe de Virasoro : aspects géométriques et algébriques, généralisations*, Publication du Centre de Recherches Mathématiques de Montréal, série "Monographies, notes de cours et Actes de conférences", PM28, 2007.

[K97] Maxim Kontsevich, *Deformation quantization of Poisson manifolds, I*, Preprint IHES, arXiv : q-alg/9709040, 1997.

[L98] Jean-Louis Loday, *Cyclic homology*, Springer-Verlag, Berlin, Heidelberg, 1998.

[P05] Anne Pichereau, *Cohomologie de Poisson en dimension trois*, C. R. Acad. Sci. Paris, Ser. I 340, 2005.

[P06] Anne Pichereau, *Poisson (co)homology and isolated singularities*, Journal of Algebra, Volume 299, Issue 2, pp. 747-777, 2006.

[RSP02] Eric Rannou, Philippe Saux-Picart, *Cours de calcul formel*, partie II, éditions Ellipses, 2002.

[RV02] Claude Roger, Pol Vanhaecke, *Poisson cohomology of the affine plane*. Journal of Algebra, Volume 251, Issue 1, pp. 448-460, 2002.

[S77] T. A. Springer, *Invariant theory*, Lecture Notes in Math., **585**, Springer-Verlag, 1977.

[VB94] Michel Van den Bergh, *Noncommutative homology of some three-dimensional quantum spaces.* Proceedings of Conference on Algebraic Geometry and Ring Theory in honor of Michael Artin, Part III (Antwerp, 1992), volume 8, pp. 213-230, 1994.